If the shock wave forms an angle $\alpha < 45°$ with the 0ξ axis, then the variation of the vertical component along 0ξ becomes smooth and less shallow than for $\alpha = 45°$.

In the last case the comparison of the results of the numerical calculation and the analytical solution [1] carried out shows that all parameters of the medium on the wave front mutually agree with an accuracy of 0.1%, while in the case of the ray approximation [5] their values are somewhat reduced. We further discover that the character of variation of the load profile along the boundary surface substantially alters the pressure distribution both in depth and along the half-space. On the basis of the analysis of the results of the calculation we note that p, u, v in the region of aftereffect of the moving load $\xi > 1$, dependent on the depth, vary according to a nonlinear law (in contrast to the region of application of the load $\xi \leqslant 1$).

From Fig. 6 we see that for $\xi = 0.2$ an exponential load in comparison with a load of finite length leads to an increase in the values of p, u, v at all points of the half-space, which was to be expected, since the value of the applied load (9) on the free surface is greater than (8).

The given scheme allows us to calculate the parameters of a nonlinearly compressed half-plane also in the case of nonlinear unloading of the medium.

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APPROXIMATE EQUATIONS OF DYNAMICS OF AN ELASTIC LAYER

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UDC 534.12

With the investigation of wave phenomena in an elastic layer, wide use has been made of approximate theories based on representation of the displacements in the form of series with respect to the middle surface [1-3]. Expansion in series in terms of Legendre polynomials is one method for the representation of the sought solution of the theory of elasticity, and has advantages over the remaining methods [1]. Retaining one number of terms or another for the coefficients of the series, different variants of the equations of the dynamics of plates can be derived. The equations of Bernoulli–Euler and Timoshenko have been the most completely investigated. In addition, for the description of processes taking place in a layer, in recent years different variants of the refined equations have been brought in [2, 4, 5].

The interest in the vibrations of plates, and the derivation of more exact equations, is connected partially with the fact that a transition from the equations of the theory of elasticity to approximate equations leads to errors in the description of non-steady-state processes. Thus, due to the approximate manner of taking account of the distribution of the displacements over the thickness of the layer, no account is taken of surface Rayleigh waves or of the fronts of waves reflected repeatedly from the surfaces of the layer; i.e., in the derivation of the equations of plates, high frequencies are ignored. These rapidly varying parts of the solution of the theory of elasticity are determined in the expanded terms of the dis-

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placements, discarded with the derivation of variants of the equations of the dynamics of plates which are sufficiently simple for analysis or for computer calculation. It is clear that a better approximation of the solution of problems of the theory of elasticity by the solutions of approximate equations can be attained by increasing the number of terms of the series and, correspondingly, increasing the order of the system of equations for a plate. Therefore, there arise questions of the errors in solution of problems of the theory of elasticity using these equations, and of the region of applicability of the equations of the dynamics of plates.

Comparisons of the solutions of approximate equations with an exact solution of the problem of elasticity have been made with large times for the long-wave asymptotic curve [6] and in the initial stage of the process for a cross section under a local load [1, 7]. It has been shown that the Kirchhoff and Timoshenko theories well describe the long-wave part of the solution; the Timoshenko theory does so in a broader range. At the start of the process, where the ratio of the thickness of the layer to the extension of the region occupied by the perturbations is on the order of unity, the rapidly varying part of the solution plays a significant role. A comparison with an exact solution of the theory of elasticity shows that here the Kirchhoff theory is unsuitable, while the Timoshenko equations give reliable results after the expansion wave has traversed a quarter of the layer thickness. The question of applicability of approximate equations in cross sections far from the point of action of a load at the moment of the arrival of perturbations remains open.

With this aim in view, in the present work a comparison is made between one exact solution of non-steady-state problem of the theory of elasticity for a layer and the equations of the dynamics of plates describing cylindrical bending. For purposes of comparison, in addition to the Bernoulli—Euler and Timoshenko equations, the article gives refined equations of the dynamics of plates. Particular stress is laid on study of the quantities in terms of which the derivatives of the displacements are expressed: the moment, the transverse force, and the rate of bending.

Let the layer occupy the region $0 \leqslant z \leqslant 1$, $|x| \leqslant \infty$, $|y| \leqslant \infty$; x, y. z are the coordinates of points in a rectangular Cartesian system. Here, the thickness of the layer h, the density of the medium, and the velocity of the expansion wave c_1 are taken as the units of measurement. A unit of measurement is the interval of time, in the course of which the expansion wave traverses a distance equal to the layer thickness.

The source of the perturbations consists of two self-equalizing loads, identical in value, applied instantaneously at the moment of time t = 0 to the layer surfaces z = 0 and z = 1. We consider a plane stressed state (the loads vary along the y coordinate)

$$\sigma_{zz} = \mp(1/2\pi)[\varepsilon/(\varepsilon^2 + x^2)]\delta_0(t), \ \sigma_{xz} = 0 \ (z = 0, \ z = 1), \tag{1}$$

where σ_{ij} are the components of the stress tensor; $\delta_0(t)$ is a function of a unit discontinuity; ε is a real parameter; — and + correspond to the planes z = 0 and z = 1. With $\varepsilon = 0$, in the boundary conditions we obtain the loads $\mp(1/2)\delta_0(t)\delta_1(x)$ concentrated along the lines x = z = 0 and x = 0, z = 1. The initial conditions of the problem are null.

This problem describes the main non-self-equalizing part of the field with bending of the layer by a vertical force acting only on the plane z = 0. The symmetrical component of the load

$$\sigma_{zz} = -(1/2\pi)[\varepsilon/(\varepsilon^2 + x^2)]\delta_0(t), \ \sigma_{xz} = 0 \ (z = 0, \ z = 1)$$

corresponds to a field which with bending has a secondary role. With a transition to the equations of the dynamics of plates, this load is determined only by the edge effect with x = 0 and the peak at the front of the shear wave $x = c_2t$. Outside these regions, the perturbations are practically equal to zero [1].

After the application of Laplace transforms (with respect to t) and Fourier transforms (with respect to x) to the equations of the theory of elasticity and to the boundary conditions (1), the transforms of the solution of the problem are found. Expanding the denominator of the transform of the solution for the layer in series in terms of exponential powers, and inverting each term, we find a solution in explicit form as the sum of repeatedly re-

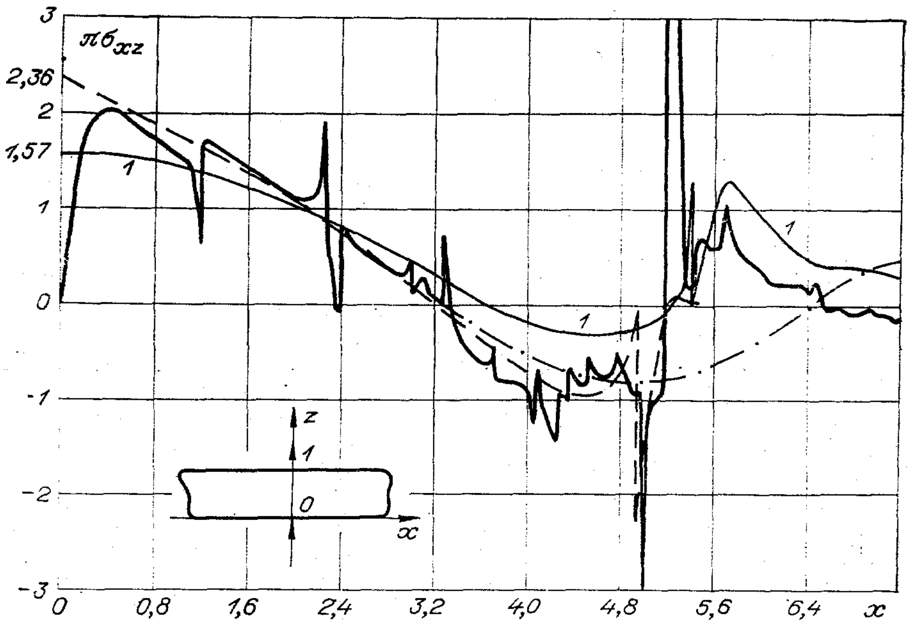

Fig. 1

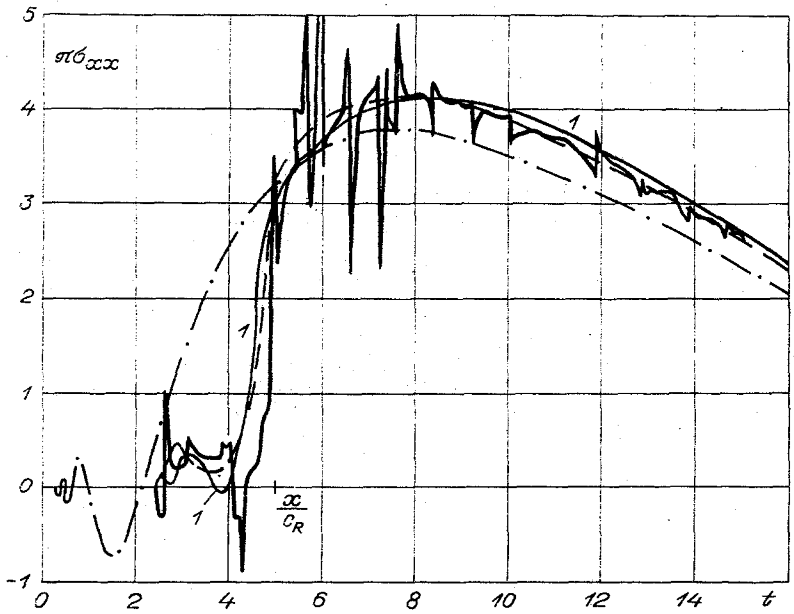

Fig. 2

flected waves. Summation and computer calculation of the analytical solution obtained were carried out in the same way as in [8]. Such a numerical—analytical method makes is possible (in distinction from a difference scheme) to calculate a rapidly varying solution with a rather high degree of exactness and to bring out the high-frequency thicknesses of the vibration, which arise as a result of the reflection of shock waves from the layer surfaces. The error in calculation of the problem is determined only by the errors in computer rounding-off in the performance of the arithmetical operations.

In Figs. 1-3, the heavy solid lines represent the results of calculations of the stresses and of the vertical component of the velocity $\partial w/\partial t$ in accordance with the theory of elasticity. The calculations were made with $\varepsilon = 0.001$; i.e., the load is concentrated with respect to the thickness of the layer. The Poisson coefficient was taken equal to 0.292. This value corresponds to a dimensionless value of the velocity of the shear wave $c_2 = 0.542$.

We note that the curves were plotted from points a distance of 0.05 apart along the axis of abscissas. Therefore, the front peaks (in their vicinity, the stresses and the velocities

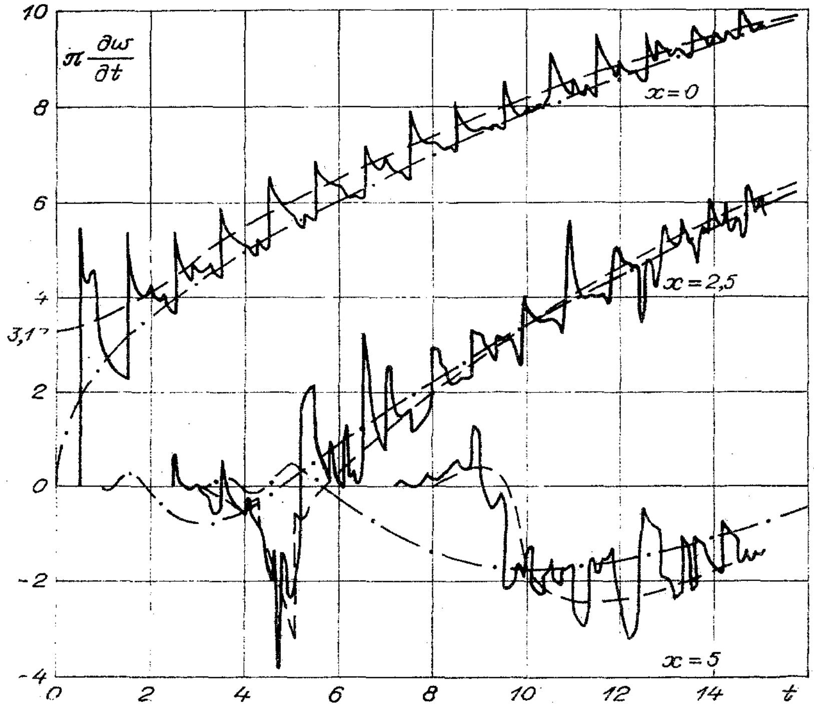

Fig. 3

are proportional to $\varepsilon^{-1/2}$) are written with a certain error, since with an increase in the time the width of a front peak, where this asymptotic is located, becomes negligibly small in comparison with the spacing 0.05.

Approximate equations of the dynamics of plates can be obtained with a given degree of exactness by expanding the displacements in series in terms of Legendre polynomials P_n in the segment $0 \leqslant z \leqslant 1$. We use the results of a derivation of the equations of the dynamics of a plate carried out in [1]. It must only be taken into account that as the unit of length here the layer thickness must be taken instead of the half-width.

In the expansions of the displacements we retain two terms

$$\begin{gathered} u(t, x, z) = 3u_1(t, x)P_1(\xi) + 7u_3(t, x)P_3(\xi), \\ w(t, x, z) = w_0(t, x)P_0(\xi) + 5w_2(t, x)P_2(\xi), \\ \xi = 2z - 1,\ 0 \leqslant z \leqslant 1. \end{gathered} \tag{2}$$

The equations of the bending deformations of a plate for the terms written in (2) with a non-self-equalizing load (1) have the form

$$\begin{gathered} L_1 u_1 - 2(1 - c_2^2)\frac{\partial w_0}{\partial x} - 4c_2^2 u(1) + 2(1 - 2c_2^2)\frac{\partial w(1)}{\partial x} = 0, \\ L_3 u_3 + 60c_2^2 u_1 - 2(1 - c_2^2)\frac{\partial w_0}{\partial x} - 10(1 - c_2^2)\frac{\partial w_2}{\partial x} - 24c_2^2 u(1) + 2(1 - c_2^2)\frac{\partial w(1)}{\partial x} = 0, \\ L_2 w_0 + 2c_2^2\frac{\partial u(1)}{\partial x} = 2\sigma_{zz}(t, x, 0), \\ L_2 w_2 + 12w_0 - 6(1 - c_2^2)\frac{\partial u_1}{\partial x} - 12w(1) + 2c_2^2\frac{\partial u(1)}{\partial x} = 0, \\ u(1) = u(t, x, 1),\ w(1) = w(t, x, 1). \end{gathered} \tag{3}$$

Here and in what follows, L_i is the operator $c_i^2(\partial^2/\partial x^2) - (\partial^2/\partial t^2)$ $(i = 0, \ldots, 3)$.

Depending on the number of terms retained in (2), different approximations of the equations of the theory of elasticity are obtained. For each approximation, there are several variants of the approximation of the solution.

Thus, neglecting in (2) the curvature of the cross section and the nonuniformity of the distribution of the bending over the thickness ($u_3 = w_2 = 0$), we obtain a first approximation, i.e., a system of equations of the fourth order for u_1, w_0 with the operator (2.11) of [1], a short-wave approximation of the solution. The equations allow us to describe the rapidly varying part of the solution, admitting an error in determination of the slowly varying components.

These shortcomings are eliminated to a considerable degree in another variant of the first approximation, i.e., the Timoshenko equations. Formally, their derivation is based on the fact that partial account is taken of the curvature of the cross sections and of the change in the value of the bending w along the coordinate z ($u_3 \neq 0$, $w_2 \neq 0$), but the derivatives of u_3 and w_2 are neglected in the second and fourth equations [1]. Expressing u_3 and w_2, after substitution into the first and third equations, we have

$$L_3 u_1 - 2c_0^2\left(6u_1 + \frac{\partial w_0}{\partial x}\right) = 0,$$
$$L_0 w_0 + 6c_0^2 \frac{\partial u_1}{\partial x} = 2\sigma_{zz}(t, x, 0), \quad c_0 = \sqrt{\frac{5}{6}}\, c_2. \tag{4}$$

In addition, neglecting the longitudinal inertia and the shift ($\partial^2 u_1/\partial t^2 \sim 0$, $6u_1 \sim -\partial w_0/\partial x$), from system (4) we obtain the Euler—Bernoulli equation

$$\frac{\partial^4 w_0}{\partial x^4} + \frac{12}{c_3^2}\frac{\partial^2 w_0}{\partial t^2} = -\frac{24}{c_3^2}\sigma_{zz}(t, x, 0),$$
$$c_3 = 2c_2\sqrt{1 - c_2^2}, \tag{5}$$

where c_3 is the rate of propagation of longitudinal perturbations in the plate.

From the procedure for the derivation of the Bernoulli—Euler equations it can be seen that the simplifications are achieved to the detriment of the rapidly varying part of the solution. Therefore, for describing the initial stage of the motion of the cross sections at the moment of the first entrances of the waves, Eq. (5) is unsuitable until the bending deformations have become established.

The Timoshenko equations are thus an intermediate variant, coinciding asymptotically with Eq. (5) for smoothly developing processes; they partly retain the advantages of the short-wave approximation.

An analogous situation is observed with the derivation of different variants of the second approximation, i.e., systems of equations of the sixth order. Retaining the values of u_1, w_0, w_2, and assuming that the cross sections remain flat with bending ($u_3 = 0$), from (3) we find a short-wave approximation of the solution. It corresponds to a system of equations of the sixth order

$$L_1 u_1 - 12c_2^2 u_1 - 2c_2^2 \frac{\partial w_0}{\partial x} + 10\left(1 - 2c_2^2\right)\frac{\partial w_2}{\partial x} = 0,$$
$$L_2 w_0 + 6c_2^2 \frac{\partial u_1}{\partial x} = 2\sigma_{zz}(t, x, 0), \tag{6}$$
$$L_2 w_2 - 60 w_2 - 6\left(1 - 2c_2^2\right)\frac{\partial u_1}{\partial x} = 0$$

with the operator [1]

$$L_1 L_2^2 - 60\left[L_2\left(L_3 - \frac{1}{5}c_2^2\frac{\partial^2}{\partial t^2}\right) + 12c_2^2\frac{\partial^2}{\partial t^2}\right].$$

This system should already give a less significant error in description of the long-wave part of the solution. The asymptotic of its operator for slow processes $(\partial^4/\partial x^4) + [(12/c_3^2)\cdot(\partial^2/\partial t^2)]$ coincides with the operator of Eq. (5).

Another variant of the refined equations in the second approximation is obtained with the condition $u_3 \neq 0$. It is assumed that the second derivatives of u_3 can be neglected in comparison with the absolute value of $168c_2^2u_3$. This permits us to express u_3 from the second equation, and to eliminate from the remaining equations of system (3)

$$L_1u_1 - 10c_2^2u_1 - \frac{5}{3}c_2^2\frac{\partial w_0}{\partial x} + \frac{5}{3}(6 - 11c_2^2)\frac{\partial w_2}{\partial x} = 0,$$

$$L_0w_0 + 5c_2^2\frac{\partial u_1}{\partial x} - \frac{5}{6}c_2^2\frac{\partial^2 w_2}{\partial x^2} = 2\sigma_{zz}(t, x, 0),$$

$$\frac{\partial^2 w_2}{\partial t^2} + 60w_2 + \frac{1}{5}\frac{\partial^2 w_0}{\partial t^2} + 6(1 - 2c_2^2)\frac{\partial u_1}{\partial x} = -\frac{2}{5}\sigma_{zz}(t, x, 0).$$

After the replacements $u_1 = \psi_*/6$, $w_0 = u_* + \chi/60$, $w_2 = \chi/60$, we obtain Eqs. (12-14) of [4], derived by a variational method.

The solution of the above variants of the equations of the dynamics of plates was sought by the method of finite differences. For Eq. (5), an explicit finite-difference scheme was adopted. The relationship between the spacing with respect to the time Δt and the spacing with respect to the spatial coordinate Δx was determined from the condition of the stability of the difference scheme for equations of the parabolic type $\Delta t = (\sqrt{3}/c_3)(\Delta x)^2$. In the calculation it was assumed that $\Delta x = 0.1$. Since $\varepsilon = 0.001$ in (1) is small and the whole change in the load takes place practically in one spacing, the value of $2\sigma_{zz}(t, x, 0)$ in the right-hand part was the difference analog of a delta-function, i.e., a rectangle with a height of $1/2 \cdot \Delta x$ and a base $2\Delta x$. As boundary conditions with $x = 0$ there were taken the absence of an intersecting force and the equality to zero of the first derivative of w_0 as a result of the symmetry of the bending with respect to the zero cross section. At infinity ($x = 600\ \Delta x$) the bending and the moment, i.e., the second derivative of w_0, were equated to zero.

With the choice of a calculating scheme for the remaining variants of the approximate equations, preference was also given to an explicit difference scheme of the "cross" type. A comparison between this and an implicit scheme showed that integration of the equations in accordance with an explicit scheme has advantages in some cases.

We note that, with integration of the approximate equations, the discontinuous part of the solution was separated out using a method developed by the authors of [9]. This was done to decrease the specific effects of a discrete model of the medium, appearing in a blurring of the discontinuities with the calculation of discontinuous solutions of hyperbolic equations. With an investigation of the prefront asymptotic of the solution of systems (4), (6), u_1 always has a discontinuity of the first kind in the second derivatives at the first and second fronts. In all cases, the derivative of w_0 had a jump only at the first front. Representing the solution in the form of the sum of the discontinuous and smooth parts, we obtain inhomogeneous systems for determining the values of the smooth (right up to the second derivative) functions u^*, w^*, v^*:

$$u^* = u_1 - u_p,\ w^* = w_0 - w_p,\ v^* = w_2 - v_p.$$

Here the discontinuous terms u_p, w_p, v_p have the following form: for system (4)

$$u_p = -\frac{0.5\,\mathrm{sgn}(x)}{c_3^2(c_3^2 - c_0^2)}\left[c_3^2(c_0t - |x|)^2\delta_0(c_0t - |x|) - c_0^2(c_3t - |x|)^2\delta_0(c_3t - |x|)\right],$$

$$w_p = 0.5c_0^{-2}(c_0t - |x|)\,\delta_0(c_0t - |x|);$$

for system (6)

$$u_p = -\frac{0.5\,\mathrm{sgn}(x)}{1 - c_2^2}\left[(c_2t - |x|)^2\delta_0(c_2t - |x|) - c_2^2(t - |x|)^2\delta_0(t - |x|)\right],$$

$$w_p = 0.5c_2^{-2}(c_2t - |x|)\,\delta_0(c_2t - |x|).$$

It was assumed that, with $x = 0$, for the components u^*, w^* there is no rotation of the cross section, and the intersecting force

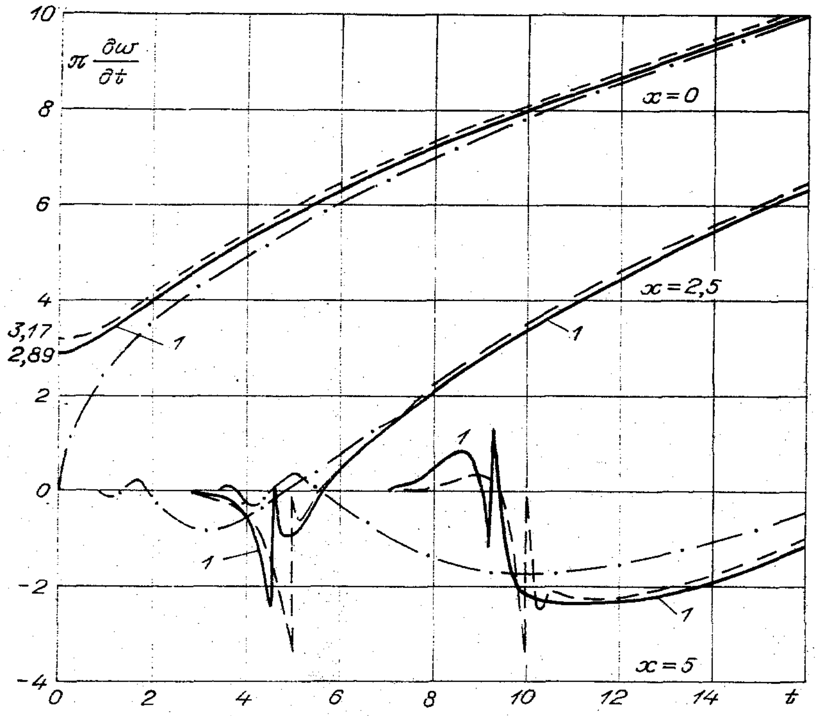

Fig. 4

$$u^* = \partial w^*/\partial x = 0$$

(x = 0). The third condition for a system of the sixth order (6) flows out of the symmetry of the component of the bending v* with respect to the zero cross section

$$\partial v^*/\partial x = 0$$

(x = 0).

Ahead of the front x = t (for the Timoshenko equations $x = c_3 t$) the displacements are equal to zero:

$$u^* = w^* = v^* = 0.$$

Some results of the calculations are given in Figs. 1-4. The dashed-dot curve corresponds to solution of the Bernoulli—Euler equations, the dashed curve to the Timoshenko equations ($\Delta t/\Delta x = 0.1$, $\Delta x = 0.025$), and curve 1 to the solution of the system (6) ($\Delta t/\Delta x = 0.999$, $\Delta x = 0.05$).

The stressed states in the cross section of the layer x = 0, calculated in accordance with approximate theories and found from the equations of the theory of elasticity, differ appreciably. Here the approximate equations correctly describe only the mean stresses over the cross section (the intersecting force, the bending moment), but the distribution of the stresses differs from the "exact" due to its considerable nonuniformity with respect to z. The approximate equations describe far better the stressed state in the other cross sections. As an illustration, Fig. 1 gives the distribution of the tangential stress along the x axis in the middle plane of the layer z = 0.5 at the moment of time t = 10. It can be seen that the Timoshenko equations (dashed) most successfully approximate the solution of the theory of elasticity in the region $x < c_R t$ (c_R is the value of the Rayleigh velocity), with the exception of a certain vicinity of x = 0. Equation (5) allows of the greatest error in description of the character of the arrival of the front of the shear wave. Equation (6) (Fig. 1, curve 1) determines shear stresses differing from the Timoshenko theory. This is connected with the fact that, in this approximation, the distribution of the shear stress over the cross section of the layer is not parabolic, but close to constant:

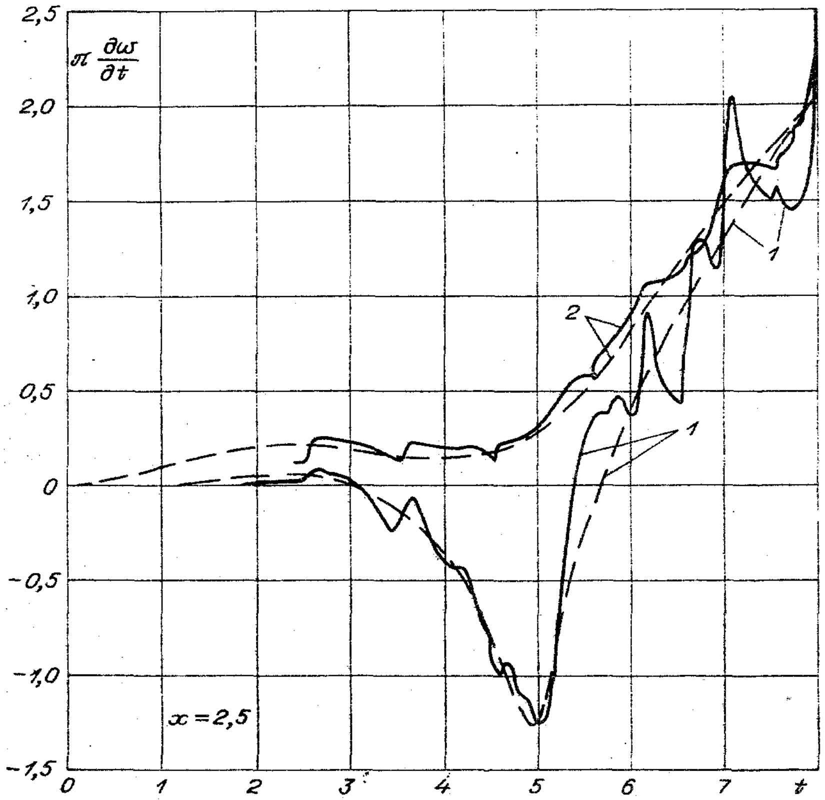

Fig. 5

$$\sigma_{xz}=c_2^2\left[6u_1+\frac{\partial w_0}{\partial x}+5\frac{\partial w_2}{\partial x}(6z^2-6z+1)\right],$$

$$\frac{\partial w_2}{\partial x}=\frac{\partial v^*}{\partial x}=0\quad(x=0).$$

The mean values of σ_{xz} over the cross section in the neighborhood of x = 0 differ only slightly in accordance with both theories. In the neighborhood of $x = c_2t$ ($5.2 < x < 6.4$), Eq. (6) describes the shear stresses better than the Timoshenko equations.

Figure 2 shows the change in σ_{xx} with time in the cross section x = 2.5 at the surface z = 0, where the load is applied. The value of x/c_R at the axis marks the time of arrival of the Rayleigh wave. It can be seen clearly how much greater is the deviation at the start of the course of the curves, determined by the Bernoulli–Euler equation (5) and by the theory of elasticity. The Timoshenko theory and the refined equations successfully describe the change in the stress in the whole interval of time. A certain deviation of the solution of Eq. (5) with large values of t is explained by the fact that the stress σ_{xx} at the surface z = 0 has not yet been established. When it becomes compressive, the solutions of the systems of equations (4), (6) go beyond the asymptotic, determined by the Eq. (5).

Figure 3 gives a representation of the development with time of the rate of bending of the middle plane of the layer z = 0.5 in the cross sections x = 0, 2.5, 5. A special characteristic in the motion of cross sections, far removed from the zero cross section, is a negative phase, observed at the start. Then, the rate becomes positive rather rapidly and rises asymptotically as $\sqrt{t}$. Under these circumstances, the solution of the theory of elasticity fluctuates around values of the rate determined by the approximate equation (5), with a frequency approximately equal to $\omega = 2\pi c_1/h$. Failure to take account of the shift and of the longitudinal inertia in the Bernoulli–Euler equation manifests itself in an earlier motion of the cross sections x = 2.5, 5 (see Fig. 3). This anticipation initially leads to considerable differences. The Timoshenko equations (dashed) reproduce the character of the appearance of perturbations fairly well. The solution using the refined equation (6) differs from the Timoshenko equations only at the moment of the arrival of the perturbations (Fig. 4, curve 1). They then come together rapidly.

With "smoothing" of the load, the solution of the Timoshenko equations is in better agreement with the solution of the theory of elasticity. As an example, Fig. 5 gives the results of calculations for loads distributed more smoothly along the x axis. The change in the signal with time is given, as before, by a Heaviside function $\delta_0(t)$. Curves 1, 2 show the dependence of the rate of bending on the time at the point x = 2.5 in the middle plane of the layer for $\varepsilon = 0.1$ and 0.5, respectively. It can be seen that, even in this small range of change in t, the Timoshenko equations (dashed line) already describe well the motion of this point. The smoother the loading, the less the differences from the exact solution.

The analysis carried out in the work allows the conclusion that the Timoshenko equations describe fairly well the reaction of a layer to the action of external bending loads. They reproduce almost exactly the character of the motion of the cross sections of the layer and the distribution of the sought values over the cross section, if it is not necessary to take account of frequencies comparable to or higher than $2\pi c_1/h$. Considerable deviations are observed only in a small neighborhood of the cross section where the local load is applied. Here reliable results can be obtained only for the rate of bending. It is not very effective to bring in refined variants of the equations of plates to describe high-frequency components of the solution of the theory of elasticity.

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